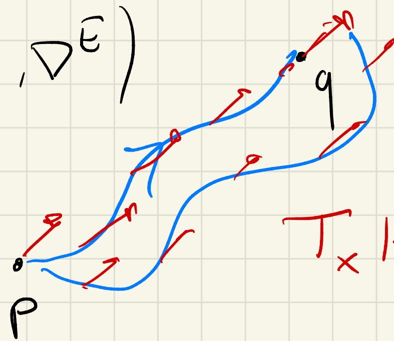
Lezione 14

(M, ∇)

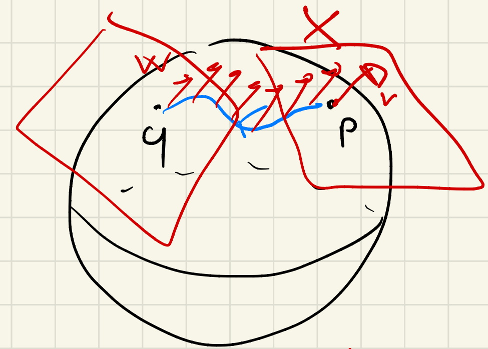
∇ → DERIVARE CAMPI LUNGO VETTORI VETTORIALI

∇ → CONNETTERE SPAZI TANGENTI

$(\mathbb{R}^n, \nabla^E)$



$T_x \mathbb{R}^n = \mathbb{R}^n$



$v \longrightarrow X \ \exists!$

parallela

$\downarrow$

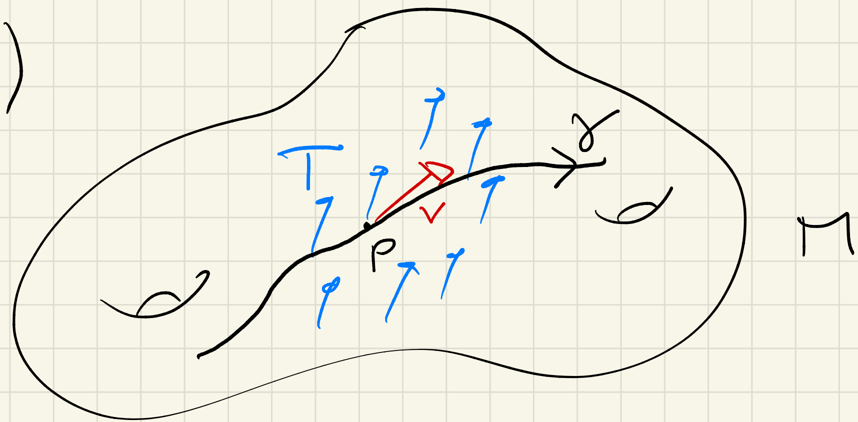
W

∇ indica anche un modo per derivare qualsiasi campo tensoriale (h,k)

in questo modo:

$$T \in \mathcal{T}^h_k(M) \qquad T(p) \in T^h_k(T_pM)$$

$$(M, \nabla)$$



Devo definire $\nabla_v T \in T^h_k(T_pM)$

Scelgo γ con $\gamma(0) = p$ $\quad \gamma'(0) = v$

$$\Gamma(\gamma)_0^t : T_{\substack{\gamma(0) \\ \parallel \\ p}} M \xrightarrow{\sim} T_{\gamma(t)} M \quad \text{isomorfismo}$$

$$\nabla_v T = \frac{d}{dt} \Gamma(\gamma)^0_t \left(T(\gamma(t))\right)\Big|_{t=0}$$

$$\Gamma(\gamma)^0_t : T^h_k\left(T_{\gamma(t)} M\right) \xrightarrow{\sim} T^h_k\left(T_{\gamma(0)} M\right)$$

(non dipende da γ)

Localmente (in coordinate):

$$T^{i_1, \dots, i_h}_{j_1, \dots, j_k} \qquad T^a{}_{bc}$$

$$v = v^i \frac{\partial}{\partial x^i} \qquad (\nabla_v T)^a{}_{bc} = v^i \frac{\partial T^a{}_{bc}}{\partial x^i} + v^i T^j{}_{bc} \Gamma_{ij}^a - v^i T^a{}_{jc} \Gamma_{ib}^j - v^i T^a{}_{bj} \Gamma_{ic}^j$$

Es: g tensore metrico

$$(\nabla_v g)_{bc} = v^i \frac{\partial g_{bc}}{\partial x^i} - v^i g_{jc} \Gamma^j_{ib} - v^i g_{bj} \Gamma^j_{ic}$$

CONNESSIONE DI LEVI-CIVITA

$$(M, g) \dashrightarrow \nabla \rightsquigarrow \text{geodetiche, curvatura}$$

TORSIONE

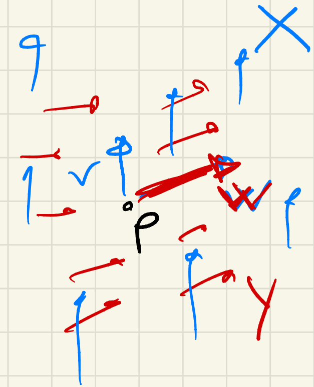
Data (M, ∇) definisco T campo tensoriale $(1,2)$

Dato $p \in M$ $\quad T(p)(v, w) \in T_pM$

$\forall v, w \in T_pM$

X, Y campi qualsiasi che estendono v e w

$$\boxed{T(p)(v,w) = \nabla_v Y - \nabla_w X - [X,Y](p)}$$



<u>Prop</u>: $T(p)(v,w)$ non dipende dalle estensioni X e Y

<u>dim</u>: In coordinate

$$T(p)(v,w) = \cancel{v^i \frac{\partial Y}{\partial x^i}} + v^i Y^j \Gamma_{ij}^k e_k - \cancel{w^i \frac{\partial X}{\partial x^i}} - w^i X^j \Gamma_{ij}^k e_k - \cancel{v^i \frac{\partial Y}{\partial x^i}} + \cancel{w^i \frac{\partial X}{\partial x^i}}$$

$$= \left(v^i w^j \Gamma_{ij}^k - w^i v^j \Gamma_{ij}^k\right) e_k \qquad \square$$

$$= \left(v^i w^j \Gamma_{ij}^k - w^j v^i \Gamma_{ji}^k\right) e_k$$

$$T(p)(v,w) = v^i w^j \underbrace{\left(\Gamma_{ij}^k - \Gamma_{ji}^k\right)}_{T_{ij}^k} e_k$$

Quindi in coordinate

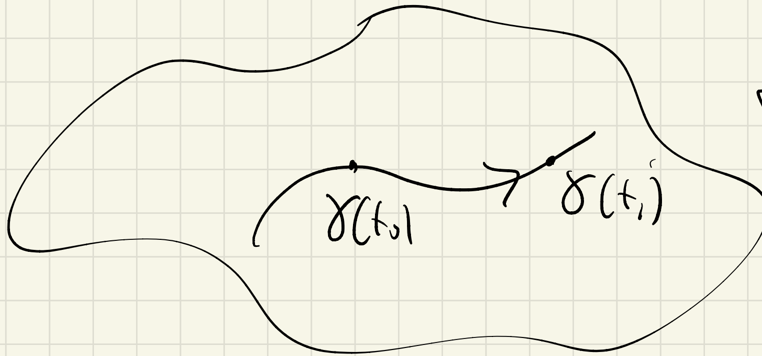
$$\boxed{T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k}$$

<u>Def</u>: ∇ è SIMMETRICA se $T \equiv 0$ cioè $T(p) = 0$ $\forall p \in M$

cioè $\Gamma_{ij}^k = \Gamma_{ji}^k$

Def: M, ∇ e g sono COMPATIBILI

se $\Gamma(\gamma)_{t_0}^{t_1} : T_{\gamma(t_0)}M \xrightarrow{\sim} T_{\gamma(t_1)}M$



è una ISOMETRIA
M (cioè che rispetti g)

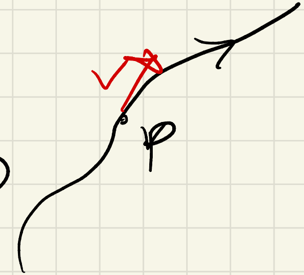
$\forall \gamma, \forall t_0, \forall t_1$

Teorema: (M, g) $\exists! \nabla$ (CONNESSIONE LEVI-CIVITA) che sia simmetrica e compatibile con g

dim.

Lemma: g è compatibile con ∇ $\Longleftrightarrow$ $\nabla_v g = 0$

$\Longleftrightarrow$ in ogni carta



$$v^i \frac{\partial g_{bc}}{\partial x^i} - v^i g_{jc}\Gamma^j_{ib} - v^i g_{bj}\Gamma^j_{ic} = 0$$

$$\forall p \quad \forall v$$

$\Longleftrightarrow$

$$\boxed{\frac{\partial g_{bc}}{\partial x^i} = g_{jc}\Gamma^j_{ib} + g_{bj}\Gamma^j_{ic}} \qquad \bigstar$$

Scrivo $\bigstar$ 3 volte permutando b, c, i

ottenete $\bigstar_A$, $\bigstar_B$, $\bigstar_C$

$$\bigstar_A + \bigstar_B - \bigstar_C \quad + \text{ SIMMETRIA } \Gamma^j_{ic} = \Gamma^j_{ci}$$

$$\frac{\partial g_{bc}}{\partial x^i} + \frac{\partial g_{ci}}{\partial x^b} - \frac{\partial g_{ib}}{\partial x^c} = g\Gamma \qquad g\Gamma$$

si ottiene

$$\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g}{\partial x} = 2g\,\Gamma$$

e infine

$$\star \qquad \boxed{\Gamma_{ij}^{\ell} = \frac{1}{2} g^{\ell k}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)}$$

Si dimostra che ∇ definita da Γ_{ij}^{l} usando ✫

è effettivamente compatibile e simmetrica

unicità in carte $\Longrightarrow$ $\exists$ in M

$(M, g) \longrightarrow \nabla \longrightarrow$ <u>geodetiche</u>

<u>Es:</u> $(\mathbb{R}^n, g^E) \longrightarrow \nabla^E$

<u>Def:</u> $\gamma : I \to M$ geodetica se $\dot{\gamma}$ è parallelo

In carte $x(t) \in \mathbb{R}^n$ è geodetica $\Longleftrightarrow$

$$\boxed{\ddot{x} + \dot{x}^i \dot{x}^j \Gamma_{ij}^{k} e_k = 0}$$

infatti in generale se $X(t)$ campo lungo $\gamma(t)$

$$D_t X = \frac{dX}{dt} + \dot{\gamma}^i X^j \Gamma_{ij}^k e_k$$

$$X = \dot{\gamma} = \dot{x}$$

γ geod $\Leftrightarrow$ $D_t X \equiv 0 \Leftrightarrow$

<u>Es:</u> ∇^E $\Gamma_{ij}^k \equiv 0$ γ geod $\Leftrightarrow$ $\ddot{x} = 0$

$\Leftrightarrow$ x è retta affine percorsa a v costante

$$x(t) = p + tv$$

$$H^2 = \{(x,y) \mid y>0\}$$

$$g = \frac{1}{y^2} g^E$$

$$\downarrow$$

$$\nabla$$

$$\Gamma^i_{jk}$$

$$\Gamma^l_{ij} = \frac{1}{2} g^{lk}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

$$\Gamma^2_{11} = \frac{1}{2} g^{2k}\left(\frac{\partial g_{1k}}{\partial x^1} + \frac{\partial g_{1k}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^k}\right)$$

$$= \frac{1}{2} g^{22}\left(2\frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right) = \frac{1}{2} y^2\left(-\frac{\partial (1/y^2)}{\partial y}\right) = \frac{1}{y}$$

$$g_{ij} = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$$

$$p = (x, y) = (x^1, x^2)$$

<u>Ex</u>: $\Gamma_{11}^{2} = \frac{1}{y} \qquad \Gamma_{12}^{1} = \Gamma_{21}^{1} = \Gamma_{22}^{2} = -\frac{1}{y}$

negli altri casi $\Gamma_{ij}^{k} = 0$

$\Gamma_{11}^{1} = 0 \ \ldots$

$$\ddot{x} + \dot{x}^i \dot{x}^j \Gamma_{ij}^{k} e_k = 0$$

$$\begin{cases} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0 \\ \ddot{y} + \frac{1}{y}\left((\dot{x})^2 - (\dot{y})^2\right) = 0 \end{cases}$$

$$i=1 \quad j=2 \qquad\qquad i=2 \quad j=1$$

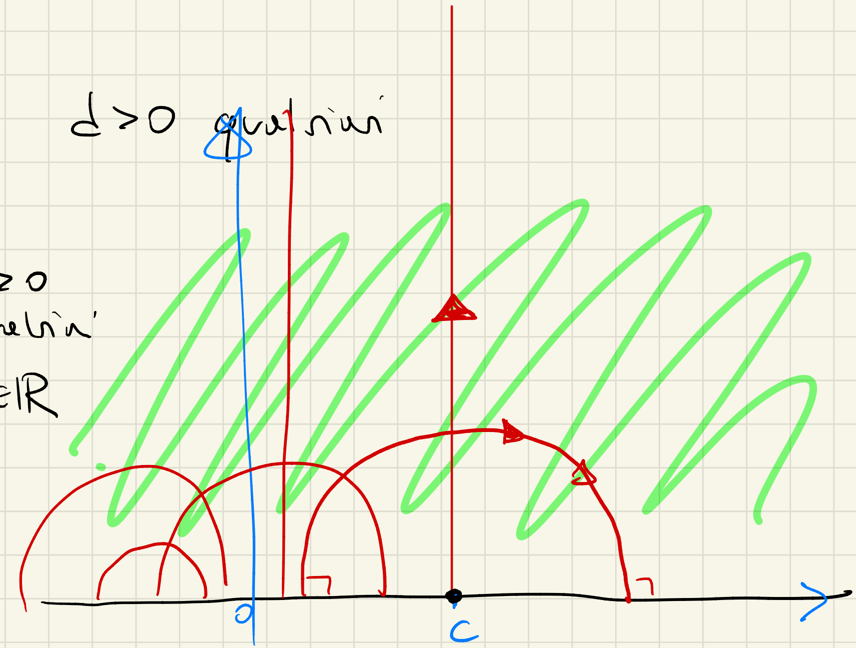
$$\ddot{x} + 2\dot{x}\dot{y}\,\Gamma_{12}^{1} = 0$$

Soluzioni: $\begin{cases} x = c \\ y = e^{dt} \end{cases}$ $\quad d>0$ qualsiasi

$$\begin{cases} x = \lambda \tanh(dt) + c \\ y = \lambda \cdot \dfrac{1}{\cosh(dt)} \end{cases}$$

$d>0$ qualsiasi

$c \in \mathbb{R}$



Oss: $(x-c)^2 + y^2 = \lambda^2$

$\Rightarrow$ geod. è un arco di circonferenza di centro $(c,0)$ e raggio λ